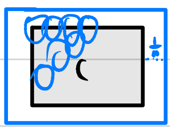
Descriptive Set Theory
Lecture 3

It's clear from the definitions of G_δ & F_σ that open $\subseteq G_\delta$ and closed $\subseteq F_\sigma$.

Proposition. In a metric space, closed sets are G_δ. Equiv., open $\subseteq F_\sigma$.

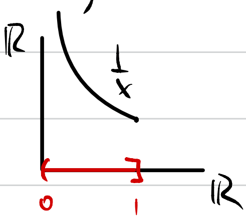
Proof. Let X be a metric space, with metric d. Let C be a closed set. Let $U_n := \{x \in X : d(x,C) < \frac{1}{n}\} = \bigcup_{y \in C} B(y, \frac{1}{n})$. Then clearly, $C \subseteq \bigcap_n U_n$.

$d(x,C) = \inf_{y \in C} d(x,y)$



Moreover, if $y \in \bigcap_n U_n$, then for each n, there is $x_n \in C$ s.t. $d(x_n, y) < \frac{1}{n}$. Then $x_n \to y$ hence $y \in C$ because C is closed. $\square$

It's trivial that closed subsets of Polish spaces are Polish. What other subsets are Polish? E.g. $(-\infty, 1]$ is Polish being a closed subset of $\mathbb{R}$. What about $(0,1]$?



The map $x \mapsto \frac{1}{x}$ is a homeomorphism of $(0,1]$ with the closed subset $\{(x, \frac{1}{x}) : x \in (0,1]\} \subseteq \mathbb{R}^2$. This suggests that open subsets are also Polish. It turns out that:

Theorem. For a Polish space X, a subset $Y \subseteq X$ is Polish iff and only if Y is G_δ.

Proof. $\Leftarrow$. First suppose that $Y = U$ is open. Then the map:

$$\sigma : x \mapsto \left(x, \frac{1}{d(x, U^c)}\right)$$

is a homeomorphism from U to a closed subset of $X \times \mathbb{R}$, where d is a complete metric for X.

To show that σ is continuous, let $x_n \to x$ and observe that $\sigma(x_n) \to \sigma(x)$. For the continuity of σ^{-1}, suppose that $\sigma(x_n) \to \sigma(x)$, and again observe $x_n \to x$.

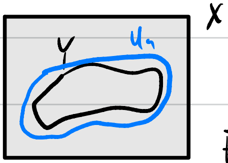
To show that $\sigma(U) = \left\{\left(x, \frac{1}{d(x,U^c)}\right) : x \in U\right\}$ is closed, use the fact that U^c is closed.

More generally, suppose $Y = \bigcap_n U_n$, $U_n \subseteq X$ open. Then the map

$$x \mapsto \left(x, \left(\frac{1}{d(x, U_n^c)}\right)_{n \in \mathbb{N}}\right)$$

is a homeomorphism of Y with a closed subset of $X \times \mathbb{R}^{\mathbb{N}}$.

$\Rightarrow$.



Let d_X, d_Y be complete compatible metrics on X and Y, respectively.

For each n, let $U_n :=$ the union of all X-open sets $B \subseteq X$ s.t.

(i) $B \cap Y \neq \emptyset$,

(ii) $\mathrm{diam}_{d_X}(B) < \frac{1}{n}$,

(iii) $\mathrm{diam}_{d_Y}(B \cap Y) < \frac{1}{n}$.

To show that $Y \subseteq \bigcap_n U_n$, fix $y \in Y$ and show that $\exists B \ni y$ satisfying (ii) & (iii). Indeed, since $B_{d_Y}(y, \frac{1}{n})$ is Y-open, $\exists$ an X-open set $V \subseteq X$ s.t. $V \cap Y = B_{d_Y}(y, \frac{1}{n})$.

Let $B := V \cap B_{d_X}(y, \frac{1}{n})$.

To show that $\bigcap_n U_n \subseteq Y$, fix $x \in \bigcap_n U_n$. Then for each n, there is $B_n \ni x$ satisfying (i)–(iii). By (i), let $y_n \in B_n \cap Y$. Since $d(x, y_n) \leq \mathrm{diam}_{d_X}(B_n) \to 0$, $y_n \xrightarrow{X} x$.

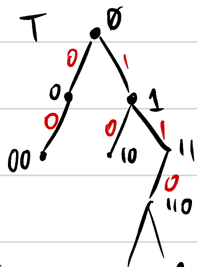
But by (iii), (y_n) is also Cauchy in d_Y, so the completeness of d_Y gives a limit $y \in Y$. Thus, topologically, $y_n \xrightarrow{X} y$ also in X. But X is Hausdorff, hence limits are unique, so $x = y$. □

Prop. $2^{\mathbb{N}}$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$, while $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a G_δ subset of $2^{\mathbb{N}}$.

Proof. In HW1. □

Trees.

Let A be a nonempty set, which we think of a set of symbols, an alphabet. A set-theoretic tree T on A is a subset of $A^{<\mathbb{N}}$ such that T is nonempty and is closed downward, i.e. if a word $w \in T$ then all of initial subwords of w are also in T, including that $\emptyset$ subword. E.g.



$A = \{0,1\}$.

T will typically be infinite, e.g. $T := A^{<\mathbb{N}}$.

This picture shows how to get a graph-theoretic rooted tree from a set-theoretic one; the converse construction is also easy.

For a tree T on A, let $[T] :=$ the set of infinite branches through $T := \{x \in A^{\mathbb{N}} : x|_n \in T\}$, where if $x = (x_i)_{i \in \mathbb{N}}$, then $x|_n := (x_i)_{i<n}$.

Prop. For any tree T on A, the set $[T]$ is closed in $A^{\mathbb{N}}$, where A is given the discrete topology.

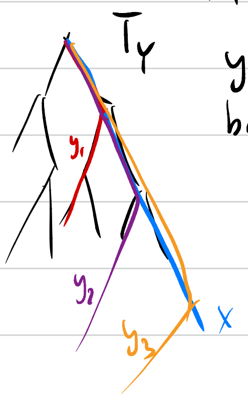
Proof. If $x_n \in [T]$ and $x_n \to x$, then $\forall m \in \mathbb{N}$, $(x_n|_m)_{n \in \mathbb{N}}$ stabilizes and equals $x|_m$, thus $x|_m \in T$. Thus, $x \in [T]$. $\square$

Now let $Y \subseteq A^{\mathbb{N}}$ and define $T_Y := \{w \in A^{<\mathbb{N}} : \exists y \in Y \ y|_{|w|} = w\} = \{y|_n : y \in Y \text{ and } n \in \mathbb{N}\}$. T_Y is a tree by definition. In fact, T_Y is a pruned tree, i.e. every $w \in T$ has an extension, i.e. $\exists a \in A$ s.t. $wa \in T$. Also by definitions, $[T_Y] \supseteq Y$.

<u>Prop.</u> A set $Y \subseteq A^{\mathbb{N}}$ is closed if and only if $Y = [T_Y]$. In particular, $Y \mapsto T_Y$ is a bijection between closed sets in $A^{\mathbb{N}}$ and pruned trees on A.

Proof. $\Leftarrow$. We already proved that $[T_Y]$ is closed.

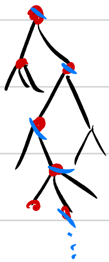
$\Rightarrow$. Suppose Y is closed and let $x \in [T_Y]$. Then $\forall n$, $x|_n \in T_Y$. Hence $\exists y_n \in Y$ s.t. $y_n|_n = x|_n$. Then $y_n \to x$ in the ptwise convergence top, so $x \in Y$ because Y is closed. $\square$



Now let's understand which pruned trees correspond to compact sets.

<u>König's lemma.</u> Every infinite finitely-branching tree T on A has an infinite branch, i.e. $[T] \neq \emptyset$.

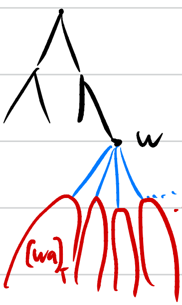
Proof. Call $w \in T$ heavy if $T_w := \{v \in T : v \supseteq w\}$ is infinite.



Then $\emptyset$ is heavy and since it has only finitely-many extensions in T, one of them has to be heavy (we're using that lightness is finitely-additive). Keep going... (technically using the Axiom of Dependent Choice). $\square$

Prop. For a pruned tree T on A, $[T]$ is compact if and only if T is finitely-branching.

Proof. $\Rightarrow$. Suppose T isn't finitely branching, then $\exists\, w \in T$ s.t. w has infinitely many extensions in T.



Then $[w]_T := \{x \in [T] : x \supseteq w\}$ is clopen subset of $[T]$ and it is covered by infinitely many disjoint clopen sets of the form $[wa]_T$, $a \in A$. Thus, $[w]_T$ isn't compact, hence $[T]$ isn't compact. (The sets $[wa]_T$ are nonempty because T is pruned.)

$\Leftarrow$. Let $\mathcal{U}$ be an open cover of $[T]$ and suppose towards a contradiction that $\nexists$ finite subcover. Define an appropriate notion of heaviness for the nodes of T to show that $\exists\, x \in [T]$ not covered by $\mathcal{U}$... (This is a proof by Jenna Zomback.)